Tetrahedron Reflection Equation. *

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Abstract

Reflection equation for the scattering of lines moving in half-plane is obtained. The corresponding geometric picture is related with configurations of half-planes touching the boundary plane in 2+1 dimensions. This equation can be obtained as an additional to the tetrahedron equation consistency condition for a modified Zamolodchikov algebra.

1 Introduction

Active recent development of the reflection equation in 1+1 dimension gives rise to variety of interesting physical results on solvable models with non-periodic boundary conditions and mathematical relations such as braid group in a solid handlebody (see [1] and Refs therein). Having this in mind and the existing of restricted, although quite elaborated, results (see [2] - [8]) for the Zamolodchikov tetrahedron equation (TE) [9], a straightforward generalization of the reflection equation for the 2+1 dimension (tetrahedron reflection equation) is given in this paper. Moving to higher dimensions we follow the kinematic analogy transforming the objects corresponding to events in smaller dimensions to generators of a Zamolodchikov algebra while defining relations for them acquire an extra exchange algebra factor; e.g. ¿from 1 dim to 2 dim we have:

$$A \cdot A \equiv A \cdot A \;\rightarrow\; A_1 \, A_2 = R_{12} \, A_2 \, A_1 \; .$$

So the commutativity condition is transformed into the Zamolodchikov (exchange) algebra. In the same manner, the Yang-Baxter equation for 2 dim is transformed into the Zamolodchikov algebra for moving straight lines on a plane $(2+1$ dimensions):

$$A^{(12)} \, A^{(13)} \, A^{(23)} = A^{(23)} \, A^{(13)} \, A^{(12)} \;\rightarrow\; A_1^{(12)} \, A_2^{(13)} \, A_3^{(23)} = R_{123} \, A_3^{(23)} \, A_2^{(13)} \, A_1^{(12)} \; ,$$

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giving rise to the tetrahedron equation as a consistency condition. This procedure can be extended for higher dimensions [10]. We note that 2-dim reflection equation (without spectral parameters) has the form of commutativity condition

$$K_1 \cdot K_{12} = K_{12} \cdot K_1 \ ,$$

where $K_{12} = R_{21}\, K_2\, R_{12}$ are dressed reflection matrices. It is remarkable that reflection equation for $2+1$ dimensions has the form of the Yang-Baxter equation but for dressed operators too (see Sect. 3).

As we will see below the geometrical picture is also helpful for the construction, where trajectories from smaller dimension started to be moving objects in higher dimension.

2 Zamolodchikov Algebra with Boundary Operators in 3 Dimensions

We recall [11] that factorizable scattering (in two-dimensional integrable quantum field theory with boundary conditions) can be described by the Zamolodchikov algebra with generators $\{A_i(x)\}$ and boundary operator B which satisfy the defining relations

$$A_i(x)\, A_j(y) = R_{ij}^{kl}(x,y)\, A_l(y)\, A_k(x) \ , \quad A_i(x)\, B = K_i^j(x)\, A_j(\overline{x})\, B \ . \tag{1}$$

The consistency conditions for this algebra give rise to the Yang-Baxter equations for matrices R

$$R_{j_1 j_2}^{i_1 i_2}(x,y)\, R_{k_1 j_3}^{j_1 i_3}(x,z)\, R_{k_2 k_3}^{j_2 j_3}(y,z) = R_{j_2 j_3}^{i_2 i_3}(y,z)\, R_{j_1 k_3}^{i_1 j_3}(x,z)\, R_{k_1 k_2}^{j_1 j_2}(x,y) \Rightarrow$$

$$R_{12}(x,y)\, R_{13}(x,z)\, R_{23}(y,z) = R_{23}(y,z)\, R_{13}(x,z)\, R_{12}(x,y) \ ,$$

and reflection equations for R and K

$$K_2(y)\, R_{12}(x,\overline{y})\, K_1(x)\, R_{21}(\overline{y},\overline{x}) = R_{12}(x,y)\, K_1(x)\, R_{21}(y,\overline{x})\, K_2(y) \ . \tag{2}$$

Here and below we use standard matrix notations of the R-matrix formalism [12]. Our aim, in this section, is to define 3d analogues of the algebra (1).

Let us consider the picture:

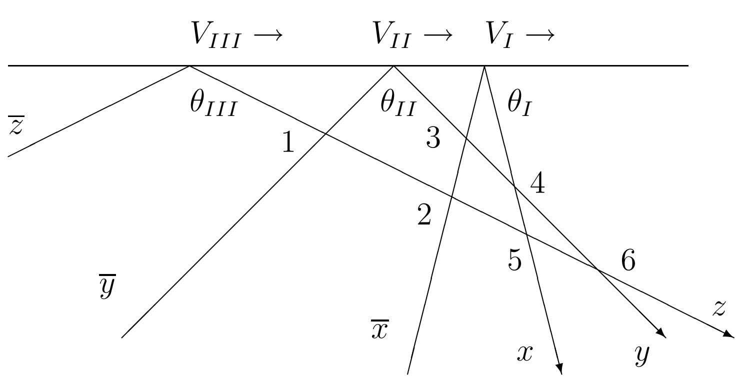


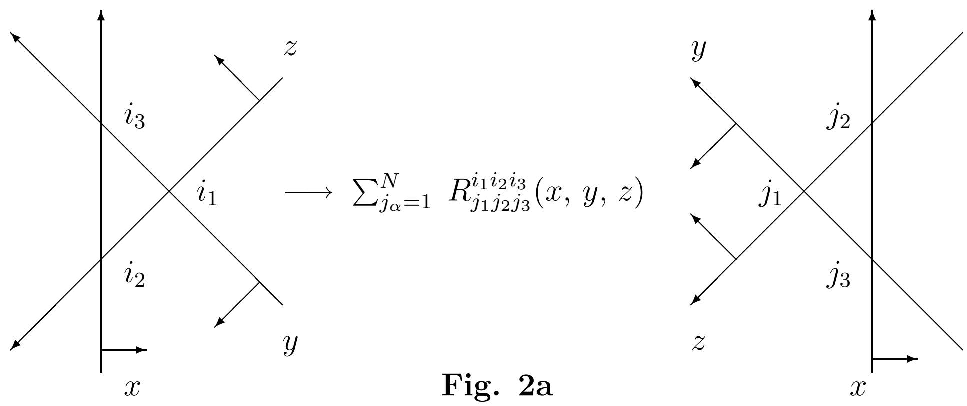
Fig. 1

We have 3 broken lines (b-lines) with peaks III, II, I moving from left to right with velocities $V_{III} > V_{II} > V_I = 0$. For convenience we will think that each b-line has 2 colours, namely segments of b-line I have colours $(x,\ \overline{x})$, b-line II has colours $(y,\ \overline{y})$ and b-line III has colours $(z,\ \overline{z})$. We denote angles between segments of b-lines as $\theta_{III,II,I}$ ($\theta_{III} > \theta_{II} > \theta_I$). One can consider colours $x,\ y,\ z$ as two-vectors (velocities of falling strings; $\overline{x},\ \overline{y}$ and $\overline{z}$ are velocities of reflecting strings) which yield another equivalent parametrization of b-lines instead of $\{V_I,\ \theta_I\}$, $\{V_{II},\ \theta_{II}\}$ and $\{V_{III},\ \theta_{III}\}$. Numbers $1, 2, \ldots, 6$ denote intersections of the b-line segments. It is clear that one can relate these numbers with pairs of colours:

$$(y,\ z) = 6\ ,\quad (x,\ z) = 5\ ,\quad (x,\ y) = 4\ ,\quad (\overline{x},\ y) = 3\ ,\quad (\overline{x},\ z) = 2\ ,\quad (\overline{y},\ z) = 1\ . \tag{3}$$

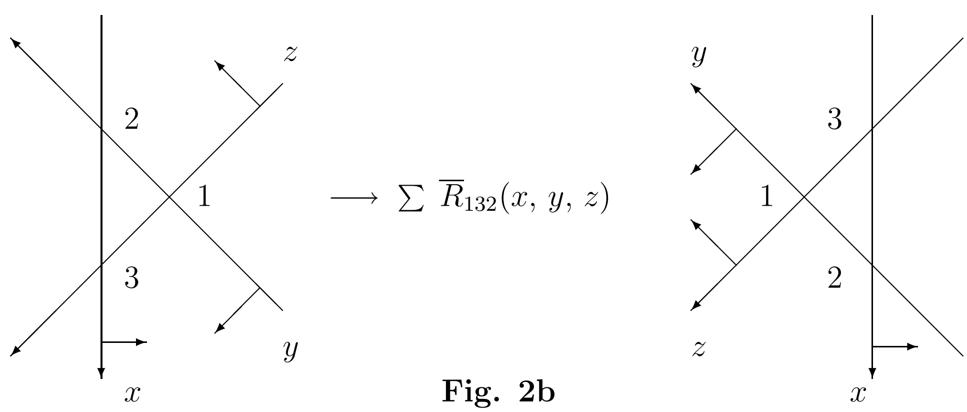
We also ascribe vector index to each intersection and, therefore, $(1, 2, \ldots, 6)$ can be considered as the numbers of vector spaces.

The following events can happen when b-lines move from left to right



$$\longrightarrow\ \sum_{j_\alpha=1}^{N}\ R^{i_1 i_2 i_3}_{j_1 j_2 j_3}(x,\ y,\ z)$$

Fig. 2a



$$\longrightarrow\ \sum\ \overline{R}_{132}(x,\ y,\ z)$$

Fig. 2b

and also (when $V_{II} > V_I$) we have

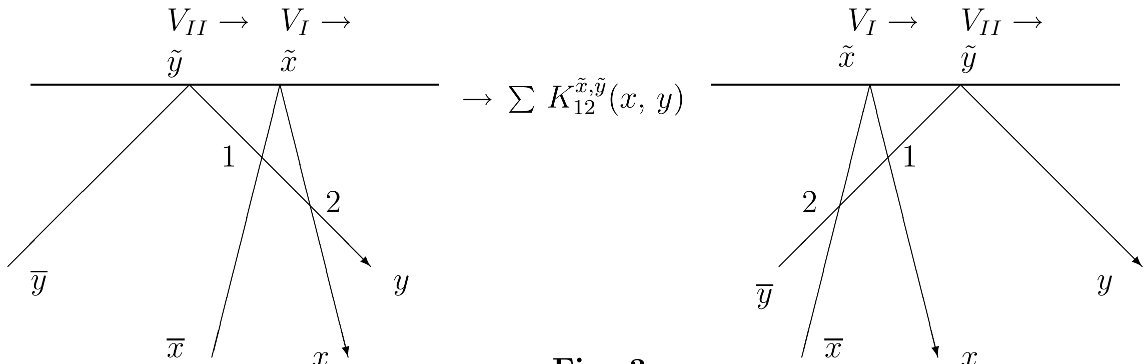


Fig. 3

Here $K_{12}^{\tilde{x},\tilde{y}}(x,\, y)$ is an operator which characterizes the reflecting of pair of strings from the border, indices $1, 2$ denote matrix spaces, x, y are two-vectors (vector velocities of falling segments of broken strings) and $\tilde{x}, \tilde{y}$ - numbers of operator spaces.

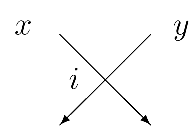
One can rewrite Figs. 2a, 2b in the algebraic form of defining relations for 3d Zamolodchikov algebra (see e.g. [13])

$$A_1^{\hat{z},\hat{y}}(z,y)\, A_2^{\hat{z},\hat{x}}(z,x)\, A_3^{\hat{y},\hat{x}}(y,x) = R_{123}(x,y,z)\, A_3^{\hat{y},\hat{x}}(y,x)\, A_2^{\hat{z},\hat{x}}(z,x)\, A_1^{\hat{z},\hat{y}}(z,y)\ , \tag{4}$$

$$A_1^{\hat{z},\hat{y}}(z,y)\, A_2^{\hat{x},\hat{y}}(x,y)\, A_3^{\hat{x},\hat{z}}(x,z) = \overline{R}_{132}(x,y,z)\, A_3^{\hat{x},\hat{z}}(x,z)\, A_2^{\hat{x},\hat{y}}(x,y)\, A_1^{\hat{z},\hat{y}}(z,y)\ , \tag{5}$$

where N- vectors (in i-th matrix space)

$$A_i^{\hat{x},\hat{y}}(x,y) \equiv$$



are generators of the 3d Zamolodchikov algebra, x, y are 2d vector- velocities, while indices $\hat{x}, \hat{y}$ denote the numbers of auxiliary operator spaces, and we omit these indices in formulas below in view of their one to one correspondence with velocities. It is clear that the kind of "unitarity" condition is valid

$$\overline{R}_{132}(x,\, y,\, z) = R_{321}^{-1}(y,\, z,\, x)\ , \tag{6}$$

and the matrices $R_{123}(x,\, y,\, z) \in (Mat(N))^{\otimes 3}$ are solutions of the tetrahedron equation

$$\begin{gathered} R_{123}(x,y,z)\, R_{145}(u,y,z)\, R_{246}(u,x,z)\, R_{356}(u,x,y) = \\ R_{356}(u,x,y)\, R_{246}(u,x,z)\, R_{145}(u,y,z)\, R_{123}(x,y,z)\ , \end{gathered} \tag{7}$$

which can be derived (see [9]) if one consider two possible ways (depending on what is the first - to turn out inside the triangle A or B) to transform the picture

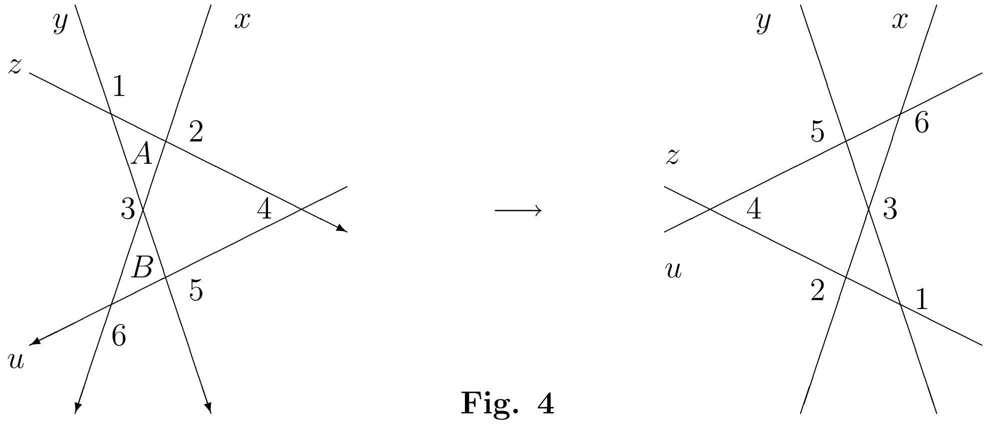


Fig. 4

or reorder corresponding monomial of sixth degree of the 3d Zamolodchikov algebra

$$A_1(z,y)\,A_2(z,x)\,A_3(y,x)\,A_4(z,u)\,A_5(y,u)\,A_6(x,u)\;.$$

Note, that the tetrahedron eq. (7) has simple solutions $R_{123}(x,y,z) = I_1\,R_{23}(y,z)$, $R_{123}(x,y,z) = R_{12}(x,y)\,I_3$, where $R_{ij}(x,y)$ is arbitrary solution of the Yang-Baxter equation. In fact, one can rewrite (7) in the following concise form

$$\begin{array}{c} R(x,y,z)\,R(u,y,z)\,R(u,x,z)\,R(u,x,y) = \\[1em] R(u,x,y)\,R(u,x,z)\,R(u,y,z)\,R(x,y,z)\;, \end{array} \tag{8}$$

since the matrix indices $1,2,\dots,6$ for R's can be restored uniquely from the order of arguments x,y,z,u. Another convenient form is

$$\begin{array}{c} R_{123}(x,y,z)\;\left[\hat{R}'_1(u,y,z)\,\hat{R}_2(u,x,z)\,\hat{R}'_3(u,x,y)\right] = \\[1em] \left[\hat{R}_3(u,x,y)\,\hat{R}'_2(u,x,z)\,\hat{R}_1(u,y,z)\right]\;R_{123}(x,y,z)\;, \end{array}$$

where $\hat{R}'_i = \mathcal{P}_{56}\,R_{i56}$, $\hat{R}_i = \mathcal{P}_{45}\,R_{i45}$ and $\mathcal{P}_{12}$ is the permutation matrix.

One can also represent Fig. 3 in algebraical form with the help of boundary operators $B^{\tilde{x}}(x)$. Namely we have for Fig. 3 the representation

$$B^{\tilde{y}}(y)\,A_1(\overline{x},y)\,B^{\tilde{x}}(x)\,A_2(y,x) = K_{12}^{\tilde{x}\tilde{y}}(x,y)\,A_2(\overline{x},\overline{y})\,B^{\tilde{x}}(x)\,A_1(\overline{y},x)\,B^{\tilde{y}}(y)\;, \tag{9}$$

which generalizes 2d reflection equation [14] (we obtain 2d reflection equation (2) if we put $K_{12} = 1$ in (9)). We stress that we try to conserve the clockwise rule in writing indices in all algebraical analogues of the pictures given above.

At the end of this discussion we present $RTTT$ relations which are 3d analogues of RTT relations (about RTT relations see [12]):

$$\begin{array}{c} R_{123}(x,y,z)\,T_1^{\hat{z},\hat{y}}(z,y)\,T_2^{\hat{z},\hat{x}}(z,x)\,T_3^{\hat{y},\hat{x}}(y,x)\;= \\ =T_3^{\hat{y},\hat{x}}(y,x)\,T_2^{\hat{z},\hat{x}}(z,x)\,T_1^{\hat{z},\hat{y}}(z,y)\,R_{123}(x,y,z)\;. \end{array} \tag{10}$$

Here $T_i(.) \in Mat_i(N)$ is the matrix in i-th matrix space. Comparing (7) with (10) one can find the matrix representations for T_i operators

$$T_1^{\hat{z}\hat{y}}(z,y) = R_{1\hat{z}\hat{y}}(u,y,z)\;,\;\; T_1^{\hat{z}\hat{y}}(z,y) = R_{\hat{z}\hat{y}1}^{-1}(y,z,u) = \overline{R}_{1\hat{z}\hat{y}}(u,y,z)\;.$$

In this section, we have obtained the algebra with generators $A_1^{\hat{x}\hat{y}}(x,y)$ and $B^{\tilde{x}}(x)$ and defining relations (4), (9). This algebra is 3d analogue of the Zamolodchikov algebra with boundary operator B (1). In the next section, we show that the consistency conditions for the algebra (4), (9) include not only tetrahedron equation (7) but also new relation for the scattering and reflection matrices R and K which can be interpreted as 3 dimensional reflection equation.

3 3d Reflection Equations

As one can see (e.g. from considering of Fig. 1) there are different scenarios of strings reflecting from the border. Namely, after the reordering of velocities $V_{III} > V_{II} > V_I$ we have 6 different choices of angles $\theta_{III} > \theta_{II} > \theta_I$, $\theta_{II} > \theta_{III} > \theta_I$ etc. It is not obvious that all of these choices give the relations which can be sewed in one 3d reflection equation. Nevertheless, below we show that all of these 6 scenarios define the unique tetrahedron reflection equation.

Scenario 1.

Here we consider two possible ways (depending on initial data) of moving b-lines on the picture Fig. 1. Namely the first way is when the top of b-line III reaches the top of b-line II, then reaches the top of b-line I and only then the top of b-line II interacts with the top of b-line I. Another way is when we have the sequence (II, I), (III, I), (III, II). The algebraic expression related to Fig. 1 (directions of arrows on strings are important here) has the form

$$1)\;\; B^{\tilde{z}}(z)\, A_1(\overline{y},z)\, B^{\tilde{y}}(y)\, A_2(\overline{x},z)\, A_3(\overline{x},y)\, B^{\tilde{x}}(x)\, A_4(y,x)\, A_5(z,x)\, A_6(z,y)\;\;, \tag{11}$$

Reordering this monomial to the new form

$$A_6(\overline{y},\overline{z})\, A_5(\overline{x},\overline{z})\, A_4(\overline{x},\overline{y})\, B^{\tilde{x}}(x)\, A_3(\overline{y},x)\, A_2(\overline{z},x)\, B^{\tilde{y}}(y)\, A_1(\overline{z},y)\, B^{\tilde{z}}(z) \tag{12}$$

(accordingly with 2 different ways described above) with the help of the rules presented on Figs. 2a, 2b, 3 and taking into account that the result should be independent of these ways (of initial data) we obtain the equation ($\theta_{III} > \theta_{II} > \theta_I$)

$$\overline{R}_{465}(z,x,y)\, R_{236}(y,z,\overline{x})\, K_{16}^{\tilde{y}\tilde{z}}(y,z)\, K_{25}^{\tilde{x}\tilde{z}}(x,z)\, \overline{R}_{153}(\overline{x},y,\overline{z})\, R_{124}(x,y,\overline{z})\, K_{34}^{\tilde{x}\tilde{y}}(x,y) =$$

$$K_{34}^{\tilde{x}\tilde{y}}(x,y)\, \overline{R}_{142}(\overline{x},z,\overline{y})\, R_{135}(x,z,\overline{y})\, K_{25}^{\tilde{x}\tilde{z}}(x,z)\, K_{16}^{\tilde{y}\tilde{z}}(y,z)\, \overline{R}_{263}(\overline{y},x,\overline{z})\, R_{456}(\overline{z},\overline{y},\overline{x})\;. \tag{13}$$

Scenario 2.

Now we take the case $\theta_y > \theta_z > \theta_x$ (here and below we use indices x, y, z instead of I, II, III). The related picture of the type Fig. 1 leads to the consideration of the monomial

$$2)\;\; A_1(\overline{z},\overline{y})\, B^{\tilde{z}}(z)\, A_2(\overline{y},z)\, B^{\tilde{y}}(y)\, A_3(\overline{x},z)\, A_4(\overline{x},y)\, B^{\tilde{x}}(x)\, A_5(y,x)\, A_6(z,x)\;,$$

which is necessary to reorder. For this we take into account relations Figs. 2a, 2b, 3 and also new relation ($V_y > V_x$)

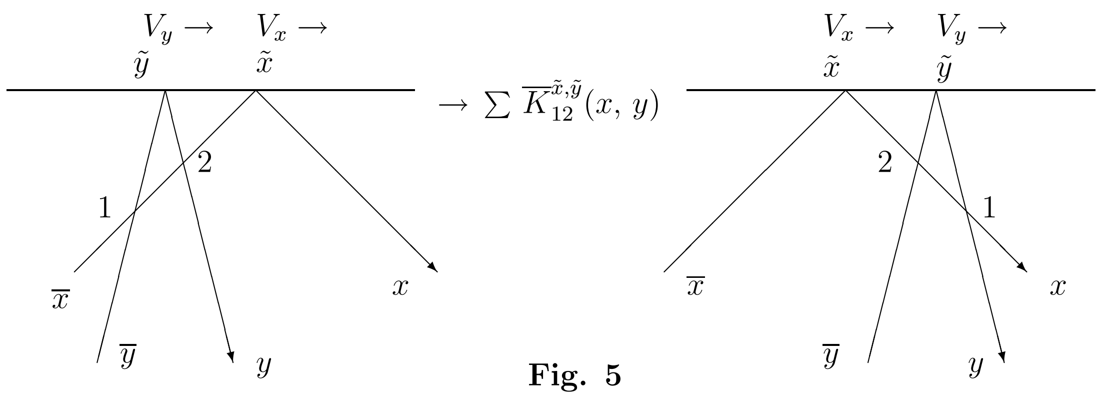


Fig. 5

which is equivalent to the algebraic formula

$$A_1(\overline{y},\overline{x})\, B^{\tilde{y}}(y)\, A_2(\overline{x},y)\, B^{\tilde{x}}(x) \;= \overline{K}_{12}^{\tilde{x}\tilde{y}}(x,y)\, B^{\tilde{x}}(x)\, A_2(\overline{y},x)\, B^{\tilde{y}}(y)\, A_1(x,y)\;. \tag{14}$$

It is evident (from comparing (9) and (14)) that the kind of unitarity condition holds

$$\overline{K}_{12}(x,y) = (K_{21}(y,x))^{-1}\;. \tag{15}$$

Now we obtain the following reflection equation

$$K_{45}^{\tilde{x}\tilde{y}}(x,y)\,\overline{R}_{253}(\overline{x},z,\overline{y})\,R_{246}(x,z,\overline{y})\,K_{36}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{165}(\overline{x},\overline{y},\overline{z})\,R_{134}(x,\overline{y},\overline{z})\,\overline{K}_{12}^{\tilde{y}\tilde{z}}(y,z) =$$

$$\overline{K}_{12}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{143}(\overline{x},z,y)\,R_{156}(x,z,y)\,K_{36}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{264}(\overline{x},y,\overline{z})\,R_{235}(x,y,\overline{z})\,K_{45}^{\tilde{x}\tilde{y}}(x,y)\;. \tag{16}$$

Scenarios 3,4,5,6.

In the same way one can consider the cases 3.) $\theta_x > \theta_y > \theta_z$, 4.) $\theta_y > \theta_x > \theta_z$, 5.) $\theta_x > \theta_z > \theta_y$ and 6.) $\theta_z > \theta_x > \theta_y$. The corresponding monomials which are necessary to reorder can be written as

3) $A_1(\overline{z},\overline{y})\, A_2(\overline{z},\overline{x})\, B^{\tilde{z}}(z)\, A_3(\overline{x},z)\, A_4(\overline{y},z)\, A_5(\overline{y},\overline{x})\, B^{\tilde{y}}(y)\, A_6(\overline{x},y)\, B^{\tilde{x}}(x)$,

4) $A_1(\overline{z},\overline{x})\, A_2(\overline{z},\overline{y})\, B^{\tilde{z}}(z)\, A_3(\overline{y},z)\, A_4(\overline{x},z)\, B^{\tilde{y}}(y)\, A_5(\overline{x},y)\, B^{\tilde{x}}(x)\, A_6(y,x)$,

5) $A_1(\overline{z},\overline{x})\, B^{\tilde{z}}(z)\, A_2(\overline{x},z)\, A_3(\overline{y},z)\, A_4(\overline{y},\overline{x})\, B^{\tilde{y}}(y)\, A_5(\overline{x},y)\, A_6(z,y)\, B^{\tilde{x}}(x)$,

6) $B^{\tilde{z}}(z)\, A_1(\overline{x},z)\, A_2(\overline{y},z)\, A_3(\overline{y},\overline{x})\, B^{\tilde{y}}(y)\, A_4(\overline{x},y)\, A_5(z,y)\, B^{\tilde{x}}(x)\, A_6(z,x)$.

The reordering of these monomials in two possible ways gives us the corresponding 3d reflection equations

$$\overline{R}_{354}(\overline{y},z,\overline{x})\,R_{125}(\overline{x},\overline{y},\overline{z})\,\overline{K}_{14}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{163}(\overline{x},z,y)\,R_{246}(y,\overline{x},\overline{z})\,\overline{K}_{23}^{\tilde{x}\tilde{z}}(x,z)\,\overline{K}_{56}^{\tilde{x}\tilde{y}}(x,y) =$$

$$\overline{K}_{56}^{\tilde{x}\tilde{y}}(x,y)\,\overline{K}_{23}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{264}(\overline{y},z,x)\,R_{136}(x,\overline{y},\overline{z})\,\overline{K}_{14}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{152}(x,z,y)\,R_{345}(y,x,\overline{z})\;. \tag{17}$$

$$\overline{K}_{23}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{254}(\overline{x},z,y)\,R_{135}(y,\overline{x},\overline{z})\,\overline{K}_{14}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{162}(y,z,x)\,R_{346}(x,y,\overline{z})\,K_{56}^{\tilde{x}\tilde{y}}(x,y) =$$

$$K_{56}^{\tilde{x}\tilde{y}}(x,y)\,\overline{R}_{364}(\overline{x},z,\overline{y})\,R_{126}(\overline{y},\overline{x},\overline{z})\,\overline{K}_{14}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{153}(\overline{y},z,x)\,R_{245}(x,\overline{y},\overline{z})\,\overline{K}_{23}^{\tilde{y}\tilde{z}}(y,z)\ . \tag{18}$$

$$\overline{R}_{243}(\overline{y},z,\overline{x})\,R_{256}(y,z,\overline{x})\,K_{36}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{164}(\overline{y},\overline{x},\overline{z})\,R_{135}(y,\overline{x},\overline{z})\,\overline{K}_{12}^{\tilde{x}\tilde{z}}(x,z)\,\overline{K}_{45}^{\tilde{x}\tilde{y}}(x,y) =$$

$$\overline{K}_{45}^{\tilde{x}\tilde{y}}(x,y)\,\overline{K}_{12}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{153}(\overline{y},z,x)\,R_{146}(y,z,x)\,K_{36}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{265}(\overline{y},x,\overline{z})\,R_{234}(y,x,\overline{z})\ . \tag{19}$$

$$\overline{K}_{34}^{\tilde{x}\tilde{y}}(x,y)\,\overline{R}_{365}(z,y,x)\,R_{246}(x,z,\overline{y})\,K_{16}^{\tilde{x}\tilde{z}}(x,z)\,K_{25}^{\tilde{y}\tilde{z}}(y,z)\,\overline{R}_{154}(\overline{y},x,\overline{z})\,R_{123}(y,x,\overline{z}) =$$

$$\overline{R}_{132}(\overline{y},z,\overline{x})\,R_{145}(y,z,\overline{x})\,K_{25}^{\tilde{y}\tilde{z}}(y,z)\,K_{16}^{\tilde{x}\tilde{z}}(x,z)\,\overline{R}_{264}(\overline{x},y,\overline{z})\,R_{356}(\overline{z},\overline{x},\overline{y})\,\overline{K}_{34}^{\tilde{x}\tilde{y}}(x,y)\ . \tag{20}$$

Now one can substitute relations (6) and (15) into eqs. (13), (16)-(20). Then we see that all these equations (written in different forms) are identical after appropriate permutation of indices $1,\ldots,6$ and velocities x,y,z. Therefore, one can explore only one of them. We take eq. (13) and substitute there (6) and definition

$$K_{12}^{\tilde{x}\tilde{y}}(x,y) \equiv \mathcal{P}_{12}\,\hat{K}_{12}^{\tilde{x}\tilde{y}}(x,y)\ .$$

As a result we obtain the following form of 3d reflection equation:

$$R_{123}^{-1}(x,y,z)\,R_{541}(y,z,\overline{x})\,\hat{K}_{16}^{\tilde{y}\tilde{z}}(y,z)\,\hat{K}_{25}^{\tilde{x}\tilde{z}}(x,z)\,R_{541}^{-1}(y,\overline{z},\overline{x})\,R_{123}(x,y,\overline{z})\,\hat{K}_{34}^{\tilde{x}\tilde{y}}(x,y) =$$

$$\hat{K}_{34}^{\tilde{x}\tilde{y}}(x,y)\,R_{456}^{-1}(z,\overline{y},\overline{x})\,R_{632}(x,z,\overline{y})\,\hat{K}_{25}^{\tilde{x}\tilde{z}}(x,z)\,\hat{K}_{16}^{\tilde{y}\tilde{z}}(y,z)\,R_{632}^{-1}(x,\overline{z},\overline{y})\,R_{456}(\overline{z},\overline{y},\overline{x})\ . \tag{21}$$

Let us put the condition $R_{123}(x,y,z) = R_{123}(\overline{x},\overline{y},\overline{z})$, which is equivalent to the conserving of special parity. Then eq. (21) is represented as

$$\begin{aligned} &\hat{K}_{54(16)32}(y,z;\overline{x})\,\hat{K}_{63(25)41}(x,z;y)\,\hat{K}_{12(34)56}(x,y;\overline{z}) = \\ &\hat{K}_{12(34)56}(x,y;z)\,\hat{K}_{63(25)41}(x,z;\overline{y})\,\hat{K}_{54(16)32}(y,z;x)\ , \end{aligned} \tag{22}$$

where we omit indices ($\tilde{x}$, $\tilde{y}$, $\tilde{z}$) and introduce dressed reflection operators

$$\hat{K}_{12(34)56}(x,y;z) = R_{123}(x,y,z)\,\hat{K}_{34}(x,y)\,R_{456}^{-1}(z,\overline{y},\overline{x})\ .$$

Examination of relation (22) leads to the conclusion that 3d reflection equation can be written in the form of the Yang-Baxter equation for dressed reflection operators.

It is tempting to investigate 3d reflection equation (21) which is independent of spectral parameters. Namely we have

$$R_{123}^{-1}\,R_{541}\,\hat{K}_{16}^{\tilde{y}\tilde{z}}\,\hat{K}_{25}^{\tilde{x}\tilde{z}}\,R_{541}^{-1}\,R_{123}\,\hat{K}_{34}^{\tilde{x}\tilde{y}} = \hat{K}_{34}^{\tilde{x}\tilde{y}}\,R_{456}^{-1}\,R_{632}\,\hat{K}_{25}^{\tilde{x}\tilde{z}}\,\hat{K}_{16}^{\tilde{y}\tilde{z}}\,R_{632}^{-1}\,R_{456}\ . \tag{23}$$

The simplest constant solution of this equation is $\hat{K}^{\tilde{x}\tilde{y}}_{12} = (I_1 \otimes I_2) \otimes (R^{\tilde{x}\tilde{y}})$, where $I_{1,2}$ are unit matrices and operators $R^{\tilde{x}\tilde{y}}$ satisfy the Yang-Baxter equation $R^{\tilde{x}\tilde{y}}\, R^{\tilde{x}\tilde{z}}\, R^{\tilde{y}\tilde{z}} = R^{\tilde{y}\tilde{z}}\, R^{\tilde{x}\tilde{z}}\, R^{\tilde{x}\tilde{y}}$. Note that one can rewrite (23) in the form (cf. with (22))

$$\hat{K}^{\tilde{y}\tilde{z}}_{54(16)32}\, \hat{K}^{\tilde{x}\tilde{z}}_{63(25)41}\, \hat{K}^{\tilde{x}\tilde{y}}_{12(34)56} = \hat{K}^{\tilde{x}\tilde{y}}_{12(34)56}\, \hat{K}^{\tilde{x}\tilde{z}}_{63(25)41}\, \hat{K}^{\tilde{y}\tilde{z}}_{54(16)32} \,. \tag{24}$$

where $\hat{K}^{\tilde{x}\tilde{y}}_{12(34)56} \equiv R_{123}\, \hat{K}^{\tilde{x}\tilde{y}}_{34}\, R^{-1}_{456}$. Moreover if we put

$$\sigma^{\tilde{x}\tilde{y}}_{12(34)56} = P_{12}\, P_{34}\, P_{56}\, \hat{K}^{\tilde{x}\tilde{y}}_{12(34)56} = P_{12}\, P_{34}\, P_{56}\, \left(R_{123}\, \hat{K}^{\tilde{x}\tilde{y}}_{34}\, R^{-1}_{456} \right) \,,$$

then we derive the relations:

$$\sigma^{\tilde{y}\tilde{z}}_{12(34)56}\, \sigma^{\tilde{x}\tilde{z}}_{54(16)32}\, \sigma^{\tilde{x}\tilde{y}}_{12(34)56} = \sigma^{\tilde{x}\tilde{y}}_{54(16)32}\, \sigma^{\tilde{x}\tilde{z}}_{12(34)56}\, \sigma^{\tilde{y}\tilde{z}}_{54(16)32} \,, \tag{25}$$

which are similar to the defining relations for the braid group.

4 Conclusion

There is a topological interpretation of the 2d reflection equation as one of the defining relation for the braid group in a solid handlebody (R^3 with an empty tube) [1], [14]. It is natural to expect that a topological interpretation exists for the 3d reflection equation as well, for the TE was connected with 2-knots in 4-dimensional space and 2-categories [16].

Another technical possibility refers to the face models, when the Boltzmann‘s weight indices are situated inside volumes of the cubic lattice. There is a transformation from the vertex models to IRC (Interaction Round Cube) models using intertwining vectors [1, 15] $\psi(\alpha; a,\, b,\, c,\, d)$. The corresponding reflection matrix (boundary Boltzmann‘s weights) is given by the relation

$$K_{12}\, \psi(1\,; d,\, e,\, f,\, c)\, \psi(2\,; f,\, g,\, b,\, c) =$$

$$= \sum_{h,\, j} Q(f,\, g|a,\, b,\, c,\, d,\, e|h,\, j)\, \psi(2\,; d,\, h,\, j,\, c)\, \psi(1\,; j,\, a,\, b,\, c) \,.$$

The indices $a,\, b,\, c,\, ...$ are inside corresponding volumes (cubes) while the intertwiner structure similar to the R-matrix one. The analogues of the relations (4), (5) for IRC models are

$$A(g \, {}^{f}_{b} \, a|z,y)\, A(g \, {}^{b}_{d} \, c|z,x)\, A(g \, {}^{d}_{f} \, e|y,x) =$$

$$= \sum_{h} W \left(g \,|\, \begin{matrix} ace \\ dfb \end{matrix} \,|\, h;\, x,y,z \right) A(h \, {}^{a}_{c} \, b|y,x)\, A(h \, {}^{e}_{a} \, f|z,x)\, A(h \, {}^{c}_{e} \, d|z,y) \,,$$

$$A(g \, {}^{f}_{b} \, a|z,y)\, A(g \, {}^{d}_{f} \, e|x,y)\, A(g \, {}^{b}_{d} \, c|x,z) =$$

$$= \sum_{h} \overline{W} \left(g \,|\, \begin{matrix} aec \\ dbf \end{matrix} \,|\, h;\, x,y,z \right) A(h \, {}^{e}_{a} \, f|x,z)\, A(h \, {}^{a}_{c} \, b|x,y)\, A(h \, {}^{c}_{e} \, d|z,y) \,,$$

where $A(a \begin{smallmatrix} d \\ b \end{smallmatrix} c|z,y) = \psi(1\,; a,\, b,\, c,\, d)\, A_1(z,y)$. Using these formulas one can rewrite (13) as tetrahedron reflection equation for IRC models. One can write 3d reflection equation with the state variables on the faces as well.

It is well known that the commutativity of the transfer matrices can be seen for the non-periodic boundary condition using two reflection equations (see [14]). Analogues procedure with peculiarities connected with TE exists for 3d reflection equation too. In particular, the important feature of the 3d RE (13) which we have to apply here is the invariance of this equation under the covariance transformation

$$K_{12}(x,y) \longrightarrow R^{-1}_{k1j}(y,\, u,\, \overline{x})\, R_{jn2}(x,\, y,\, u)\, K_{12}(x,y)\, R^{-1}_{2ki}(u,\, \overline{y},\, \overline{x})\, R_{i1n}(x,\, u,\, \overline{y})\ ,$$

where u is arbitrary 2-vector and i, j, k, n are indices of auxiliary matrix spaces.

An algebraic foundation of the TE (as quantum groups for the Yang-Baxter equations) is not completely clarified up to now (see however [17]). We think that the new 3d algebras presented in this paper will help in understanding of the algebraic structures lying behind the tetrahedron and 3d reflection equations.

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